Lecture I - Motion in One Dimension

Definitions

- Displacement: $\Delta x \equiv x_f - x_i$
- Average velocity: $\bar{v}_x = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$
- Average speed: $\frac{\text{total distance}}{\text{total time}}$

Velocity and speed NOT interchangeable in physics!

Instantaneous Velocity

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Tangent Line ; Derivative

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Average Acceleration

$$\bar{a}_x \equiv \frac{\Delta v_x}{\Delta t} = \frac{v_{xf} - v_{xi}}{t_f - t_i}$$

Instantaneous Acceleration

$$a_x \equiv \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

Integration

$$\Delta x = x_f - x_i = \lim_{\Delta t_n \to 0} \sum_n v_{xn} \Delta t_n = \int_{t_i}^{t_f} v_x(t)\, dt$$

Kinematic Equations

$$a_x = \frac{dv_x}{dt} \Rightarrow dv_x = a_x dt$$

$$v_x(t) = \int_{t_i}^{t} a_x(t')\, dt' + C_1$$

$$t = t_i,\ v_{xi} \equiv v_x(t_i) = C_1$$

$$v_x(t) = v_{xi} + \int_{t_i}^{t} a_x(t')\, dt'$$

$$v_x = \frac{dx}{dt} \Rightarrow dx = v_x dt$$

$$x(t) = \int_{t_i}^{t} v_x(t')\, dt' + C_2$$

$$t = t_i,\ x_i \equiv x(t_i) = C_2$$

$$x(t) = x_i + \int_{t_i}^{t} v_x(t')\, dt' = x_i + v_{xi}(t - t_i) + \int_{t_i}^{t} dt' \int_{t_i}^{t'} dt''\, a_x(t'')$$

Constant Acceleration

Set $t_i = 0,\ t_f = t,\ v_{xf} = v_{xi} + a_x t$

$$\bar{v}_x = \frac{v_{xi} + v_{xf}}{2}$$

$$x_f - x_i = \frac{1}{2}(v_{xi} + v_{xf})t = v_{xi}t + \frac{1}{2}a_x t^2 = \frac{v_{xf}^2 - v_{xi}^2}{2a_x}$$

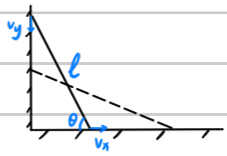
Significant Figures

- Physical quantities measured are known only to within the limits of the experimental uncertainty.

- Quality of apparatus
- Skill of the experimenter
- Number of measurements performed

- Significant figures
  - Example: $(5.5 \pm 0.1)\,\text{cm} \times (6.4 \pm 0.1)\,\text{cm} = (35 \pm 1)\,\text{cm}^2$
- Zero may be significant
  - Example: $1.5\,\text{kg} \neq 1.50\,\text{kg} \neq 1.500\,\text{kg}$

Example 1



Sol. $\ell^2 = x^2 + y^2$

Differentiate both sides with respect to t: $0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow 2xv_x + 2yv_y = 0$

$\Rightarrow \frac{v_x}{v_y} = -\frac{y}{x} = -\tan\theta$

Lecture II – Motion in High Dimensions

Choosing Different Axes

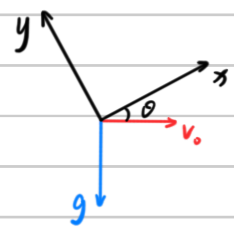
$v_x = v_0 \cos\theta - gt\sin\theta$

$x = x_0' + v_0 t\cos\theta - \frac{1}{2}gt^2\sin\theta$

$v_y = -v_0\sin\theta - gt\cos\theta$

$y = y_0' - v_0 t\sin\theta - \frac{1}{2}gt^2\cos\theta$

It needs the concept of matrix to represent the rotation.



The More Educated Way

Using vectors to unify the formulas:

- $\vec{v} = \vec{v}_0 + \vec{a}t$
- $\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$

Finite Rotations are Not Vectors

The commutative law of addition is not satisfied by finite rotations.

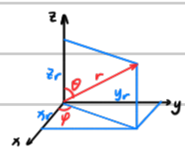
Not everything with a magnitude and a direction is a vector!

Polar Coordinates (Spherical System)

$x_r = r\sin\theta\cos\varphi$

$y_r = r\sin\theta\sin\varphi$

$z_r = r\cos\theta$



Kinamatics in Vector Description

- The motion of a particle (moving in high dimentions) is completely known if its position vector $\vec{r}$ is known as a function of time.
- Instantaneous Velocity: $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$

  $d\vec{r}$ and $\vec{r}$ are not in the same direction!
- Acceleration Vector: $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

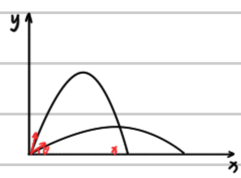
Projectile Motion

$v_x = v_0\cos\theta$, $v_{y_0} = v_0\sin\theta$

$gt = 2v_{y_0}$

$x = v_x t$

Solve equations simultaneously to obtain $x = \frac{v_0^2 \sin 2\theta}{g}$

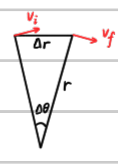


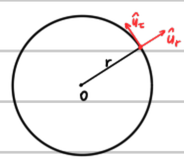
Uniform Circular Motion

$\vec{v} = v\hat{u}_t$

$\vec{a} = -\frac{v^2}{r}\cdot\hat{r}$ (centripetal acceleration)

Proof $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{u}_t) = v\frac{d\hat{u}_t}{dt} = -v\frac{d\theta}{dt}\hat{u}_r = -v\omega\hat{u}_r = -\frac{v^2}{r}\hat{u}_r$



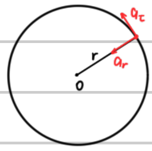


Non-uniform Circular Motion

$\vec{v} = v\hat{u}_\tau$

$\vec{a}_r = -\frac{v^2}{r}\cdot\hat{r}$, $\vec{a}_\tau = \frac{dv}{dt}\hat{u}_\tau$

$\vec{a} = a_r\hat{u}_r + a_\tau\hat{u}_\tau$ $a = \sqrt{a_r^2 + a_\tau^2}$



Lecture III — Newton's Laws

Fluid Resistance

The magnitude of the resistance force can depend on speed in a complex way.

(i) For objects falling slowly through a liquid and for very small objects (e.g., dust particles moving through air):

$R = bv$

(ii) For large objects (e.g. skydiver moving through air in free fall):

$R = cv^2$

Kinematics Analysis for Fluid Resistance

(i) At low speed

Differential equation: $mg - bv = ma = m\frac{dv}{dt}$

Static solution: $v_t = \frac{mg}{b}$

Characteristic time constant $\tau = \frac{m}{b} = \frac{v_t}{g}$

$v(t) = \frac{mg}{b}(1 - e^{-\frac{bt}{m}}) = v_t(1 - e^{-\frac{t}{\tau}})$

$a(t) = ge^{\frac{t}{\tau}}$

Proof $mg - bv = m\frac{dv}{dt} \Rightarrow \frac{dv}{g - \frac{b}{m}v} = dt$

Integrate both side: $-\frac{m}{b}\ln|g - \frac{b}{m}v| = t + C \Rightarrow v = \frac{m}{b}(g - Ke^{-\frac{b}{m}t})$

Substitute $v_i = 0$, we obtain $K = g$

Thus $v(t) = \frac{mg}{b}(1 - e^{-\frac{t}{\tau}})$

$a(t) = \frac{dv}{dt} = ge^{\frac{t}{\tau}}$

Note $v(\tau) = v_t(1 - e^{-1})$, that is how we define τ.

For a first-order linear differential equation in kinematics, its standard form is given by: $Av + B = \frac{dv}{dt}$.

In this case, the time constant is always equal to the reciprocal of the coefficient of v, that is: $\tau = \frac{1}{A}$

(ii) At high speed

Resistance force: $R = \frac{1}{2}D\rho Av^2$ D: drag cofficient ρ: density of fluid A: cross-sectional area of the falling object

Static solution: $v_t = \sqrt{\frac{2mg}{D\rho A}}$

Proof $mg - \frac{1}{2}D\rho Av^2 = ma$

Substitute $a_t = 0$, we obtain $v_t = \sqrt{\frac{2mg}{D\rho A}}$

Lecture IV - Work and Kinetic Energy

Work Done by a Varying Force

$W = \int_{x_i}^{x_f} F_x \, dx$

Hooke's Law

The force law for springs (in the limiting case of small displacements):

$F_s = -kx$ x: the displacement of the block from its unstretched ($x=0$) position

k: the force constant of the spring

Work Done by a Spring

$W_s = \int_{x_i}^{x_f} F_s \, dx = \int_{-x_{max}}^{0} (-kx) \, dx = \frac{1}{2} k x_{max}^2$

Kinetic Energy

Consider an object moving under a constant net force:

$d = \frac{1}{2}(v_i + v_f)t$, $a = \frac{v_f - v_i}{t}$

$\sum W = (\sum F) d = mad = m\left(\frac{v_f - v_i}{t}\right) \cdot \frac{1}{2}(v_i + v_f)t = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$

Thus we define the kinatic energy: $K = \frac{1}{2} m v^2$

Work-Kinetic Energy Theorem

$\sum W = K_f - K_i = \Delta K$

Proof $\sum W = \int_{x_i}^{x_f} (\sum F_x) \, dx = \int_{x_i}^{x_f} m a_x \, dx$

$a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$

$\sum W = \int_{x_i}^{x_f} mv \frac{dv}{dx} \, dx = \int_{v_i}^{v_f} mv \, dv = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$

Power

Average power: $\overline{P} = \frac{W}{\Delta t}$

Instantaneous power: $P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} = \vec{F} \cdot \vec{v}$

Kinetic Energy at High Speeds

Newtonian mechanics valid only for particle motion at small speeds ($v \ll c$).

Therefore, the truth situation at high speed should be:

$K = mc^2 \left(\frac{1}{\sqrt{1 - (\frac{v}{c})^2}} - 1 \right)$

Lecture V - Conservation of Energy

Elastic Potential Energy

$W_s = \int_{x_i}^{x_f} (-kx)\,dx = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$

Thus, the elastic potential energy associated with the system is defined by $U_s = \frac{1}{2}kx^2$

Then $W_s = U_i - U_f = -\Delta U_s$

Gravitational Potential Energy

$W_g = (m\vec{g})\cdot d = (-mg\hat{y})\cdot(y_f - y_i)\hat{y} = mgy_i - mgy_f$

Thus, we difine the gravitational potential energy of a particle, which has mass m and is at a distance y above the zero-potential-energy surface, is $U_g = mgy$

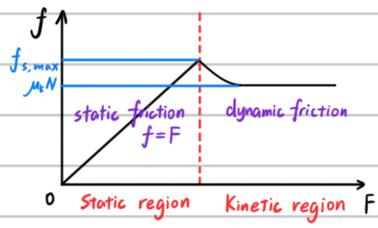
Then $W_g = U_i - U_f = -\Delta U_g$

Note Given that how we define g, the formula is only valid for objects near the surface of the Earth.

Frictional Force

Experimentally, we find that, to a good approximation, both $f_{s.max}$ and f_k are proportional to the normal force acting on the object.

It can be seen that $f_{s.max} \geq \mu_k N$



Work Done by Frictional Force

$W_f = \vec{f_k}\cdot\vec{d} = -f_k d$

Conserved Force

A force is conservative if the work it does on a particle moving between any two points is independent of the path taken by the particle.

$W_c = U_i - U_f = -\Delta U$

In contrast, a force is nonconservative if the work it does on a particle moving between any two points is dependent of the path taken by the particle.

Based on the definition, we can find that the elastic force and gravitational force are conservative, the frictional force is nonconservative.

Conservation of Mechanical Energy

The total mechanical energy of a system remains constant in any isolated system of objects that interact only through conservative forces.

$E \equiv K + U$

Derivative of Potential Energy

If the potential energy of the system is known, then $W = F_x \Delta x = -\Delta U$

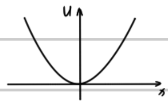
Thus, $F_x = -\frac{dU}{dx}$

Example $F_s = -\frac{dU_s}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$

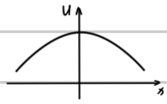
$F_g = -\frac{dU_g}{dy} = -\frac{d}{dy}(mgy) = -mg$

Equilibrium of a System

In general, positions of stable equilibrium correspond to points for which $U(x)$ is a minimum.
$\frac{dU}{dx} = 0$, $\frac{d^2U}{dx^2} < 0$



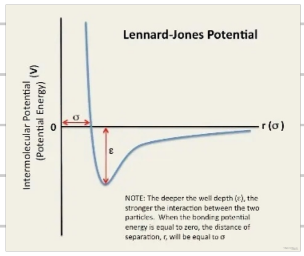
In contrast, positions of unstable equilibrium correspond to points for which $U(x)$ is a maximum.
$\frac{dU}{dx} = 0$, $\frac{d^2U}{dx^2} > 0$



Lennard-Jones Potential

The potential energy associated with the force between two neutral atoms in a molecule can be modeled by the Lennard-Jones potential energy function:

$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$



Lecture VI - Conservation of Momentum

Impulse - Momentum Theorem

The impulse of the force F acting on a particle equals the change in the linear momentum of the particle caused by that force $I = \Delta p$.

Linear momentum $\vec{p} = m\vec{v}$

Impulse $\vec{I} = \int_{t_i}^{t_f} \vec{F} dt$

Conservation of Linear Momentum

Whenever two or more particles is an isolated system interact, the total momentum of the system remains constant.

$\vec{p}_{tot} = \sum_{system} \vec{p}_i = constant$

The only requirement is that the forces must be internal to the system.

Elastic and Inelastic Collision

Momentum is conserved in any collision in which external forces are negligible.

Kinetic energy may or may not be constant.

- Elastic collision between two objects is one in which total kinetic energy is the same before and after the collision.
- Inelastic collision is one in which total kinetic energy is not the same before and after the collision.

Perfectly Inelastic Collisions

When the colliding objects stick together after the collision, the collision is called perfectly inelastic.

$m_1\vec{v}_{1i} + m\vec{v}_{2i} = (m_1+m_2)\vec{v}_f \Rightarrow \vec{v}_f = \frac{m_1\vec{v}_{1i} + m\vec{v}_{2i}}{m_1+m_2}$

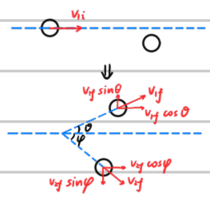
Elastic Collision in One Dimension

$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$

$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

$\Rightarrow v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1+m_2} v_{2i}, \quad v_{2f} = \frac{2m_1}{m_2+m_1} v_{1i} + \frac{m_2 - m_1}{m_2+m_1} v_{2f}$

Elastic Collision in Two Dimension



$m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\varphi$

$0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\varphi$

$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

Note Generally, we need one more equation to solve.

Special Case: Equal mass

If $m_1 = m_2$, then

$v_{1i}^2 = v_{1f}^2 + v_{2f}^2, \quad \vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$

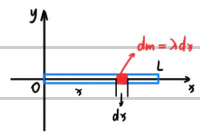
$\Rightarrow v_{1i}^2 = (\vec{v}_{1f} + \vec{v}_{2f})^2 = v_{1f}^2 + v_{2f}^2 + 2\vec{v}_{1f}\cdot\vec{v}_{2f} \Rightarrow \vec{v}_{1f}\cdot\vec{v}_{2f} = 0 \Rightarrow \theta + \varphi = \frac{\pi}{2}$

The Centre of Mass

Consider a system of many particles: $\vec{r}_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_i \cdot r_i}{M} = \frac{1}{M}\int \vec{r}\, dm$

Thus, in each dimension, we have $x_{cm} = \frac{1}{M}\int x\, dm$, $y_{cm} = \frac{1}{M}\int y\, dm$, $z_{cm} = \frac{1}{M}\int z\, dm$

The Centre of Mass of a Rod



Show that the centre of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

$$x_{cm} = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\lambda}{M}\cdot\frac{L^2}{2} = \frac{L}{2}$$

Note The centre of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.

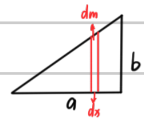
The Centre of Mass of a Nonuniform Rod

Suppose a rod is nonuniform such that its mass per unit length varies linearly with x according to $\lambda = \alpha x$.

$$M = \int dm = \int_0^L \lambda\, dx = \int_0^L \alpha x\, dx = \frac{\alpha L^2}{2}$$

$$x_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{1}{M}\int_0^L \alpha x^2\, dx = \frac{\alpha L^3}{3M} = \frac{2}{3}L$$

The Centre of Mass of a Right Triangle



$$dm = \frac{M}{\frac{1}{2}ab}\cdot y\, dx = \frac{2My}{ab}\, dx$$

$$x_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^a x\cdot\frac{2My}{ab}\, dx = \frac{2}{ab}\int_0^a xy\, dx = \frac{2}{ab}\int_0^a x\cdot\left(\frac{b}{a}x\right) dx = \frac{2}{3}a$$

Similarly, $y_{cm} = \frac{1}{3}b$

Motion of a Many-Particles System

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{M}$$

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{\sum_i m_i \vec{v}_i}{M}$$

$$M\vec{v}_{cm} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{tot}$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{\sum_i m_i \vec{a}_i}{M}$$

$$M\vec{a}_{cm} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i$$

Konig Theorem: Kinetic Energy in Centre of Mass Frame

$$E_k = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i (v_{relative} + v_{cm})^2$$

$$= \sum_i \frac{1}{2}m_i v_{relative}^2 + v_{cm}\sum_i m_i v_{relative} + \sum_i \frac{1}{2}m_i v_c^2$$

$$= \frac{1}{2}\sum_i m_i v_{relative}^2 + \frac{1}{2}M v_{cm}^2$$

$$= E_{cm} + \sum_i E_{k,\,relative}$$

Show that: Total kinetic energy of the system is made of by two parts - the kinetic energy of motion about the centre of mass, the kinetic energy with all the mass concentrated at the centre of mass.

Lecture VII - The Law of Gravity

Newton's Law of Universal Gravitation

$F_g = G\frac{m_1 m_2}{r^2}$

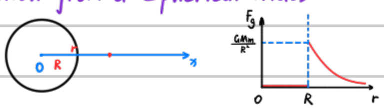
The Inverse-Square Law

$T^2 = \frac{4\pi^2 R^3}{GM}$

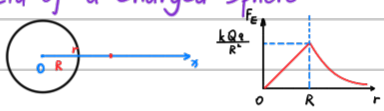
Proof $m\frac{v^2}{R} = \frac{GMm}{R^2}$, $v = \frac{2\pi R}{T} \Rightarrow T^2 = \frac{4\pi R^3}{GM}$

Note For a given astronomical object, $\frac{T^2}{R^3} = \frac{4\pi^2}{GM} = \text{constant}$, which is known as Kepler's Third Law.

Attraction from a Spherical Mass



E-Field of a Charged Sphere



Kepler's Laws

- All planets move in elliptical orbits with the Sun at one focal point.
- The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals. (The conservation of angular momentum)
- The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

The Density of the Earth

$\rho_E = \frac{3g}{4\pi G R_E}$

Proof $mg = \frac{GM_E m}{R_E^2} \Rightarrow M_E = \frac{gR_E^2}{G}$

$V_E = \frac{4}{3}\pi R_E^3$

$\rho_E = \frac{M_E}{V_E} = \frac{3g}{4\pi G R_E}$

Satellite Orbit

Total mechanical energy is conserved: $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$

$\frac{GMm}{r^2} = ma = \frac{mv^2}{r} \Rightarrow E = -\frac{GMm}{2r}$

The Minimum Speed to Launch a Satellite

$\frac{1}{2}mv^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{2R_E} \Rightarrow v = \sqrt{\frac{GM_E}{R_E}}$

Changing the Orbit of a Satellite

$\Delta E = E_f - E_i = -\frac{GM_E m}{2}\left(\frac{1}{R_f} - \frac{1}{R_i}\right)$

Escape Speed from the Earth

$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{max}}$$

$$v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right)$$

when $r_{max} \to +\infty$, $v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$

Rocket Propulsion

$$(M+\Delta m)v = M(v+\Delta v) + \Delta m(v - v_e) \Rightarrow M\Delta v = v_e \Delta m \Rightarrow M dv = v_e dm = -v_e dM$$

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M} \Rightarrow v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

Note: Under such circumstance, the momentum of the system is conserved, but the kinetic energy of the system increases (at the expense of chemical potential energy in the fuel).

Thrust of a Rocket

The thrust on the rocket is the force exerted on it by the ejected exhaust gases.

$$I = M\frac{dv}{dt} = \left|v_e \frac{dM}{dt}\right|$$

The thrust increases as the exhaust speed increases and as the rate of change of mass (called the burn rate) increases.

Lecture VIII - Rotation of a Rigid Object about a Fixed Axis

Angular Displacement and Velocity

Angular displacement: $\Delta\theta = \theta_f - \theta_i$

Average angular velocity: $\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$

Instantaneous angular velocity: $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

Average angular acceleration: $\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$

Instantaneous angular acceleration: $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$

Note We use the Right-Hand Rule to determine the directions of ω and α, which are along the axis.

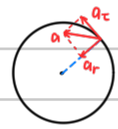
Rotational Kinematics

- $\omega_f = \omega_i + \alpha t$
- $\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$
- $\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$

Relationship between Linear Velocity and Angular Velocity

$v = \frac{ds}{dt} = r \cdot \frac{d\theta}{dt} \Rightarrow v = r\omega$

Relationship between Linear Acceleration and Angular Acceleration

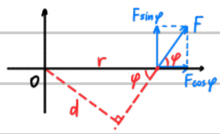


$a_\tau = \frac{dv}{dt} = r \cdot \frac{d\omega}{dt} \Rightarrow a_\tau = \alpha r$

$a_r = \frac{v^2}{r} = \omega^2 r$

$a = \sqrt{a_\tau^2 + a_r^2} = \sqrt{\alpha^2 r^2 + \omega^4 r^2} = r\sqrt{\alpha^2 + \omega^4}$

Torque



$\tau = rF\sin\varphi = Fd$, $\vec{\tau} = \vec{r} \times \vec{F}$

Here, d is called the moment arm of F.

Torque is defined only when a reference axis is specified. Its sign is determined by the direction of the force, being positive when counterclockwise and negative when clockwise.

Moment of inertia

$I = \sum_i m_i r_i^2$

The moment of inertia is a measure of the resistance of an object to changes in its rotational motion. It only depends on the physical arrangement of that mass.

Torque and Angular Acceleration

$dF = (dm)a_\tau \Rightarrow d\tau = r\,dF_\tau = (r\,dm)a_\tau$

Substitute $a_\tau = \alpha r$, then $d\tau = (r\,dm)(\alpha r) = (r^2 dm)\alpha$

$\sum \vec{\tau} = \int (r^2 dm)\alpha = \alpha \int r^2 dm = I\alpha$

Rotational Kinetic Energy

$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2} I\omega^2$

Work and Power in Rotation

Work: $dW = \vec{F}\cdot d\vec{s} = (F\sin\varphi)(r\,d\theta) = \tau\,d\theta$

Instantaneous Power: $P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$

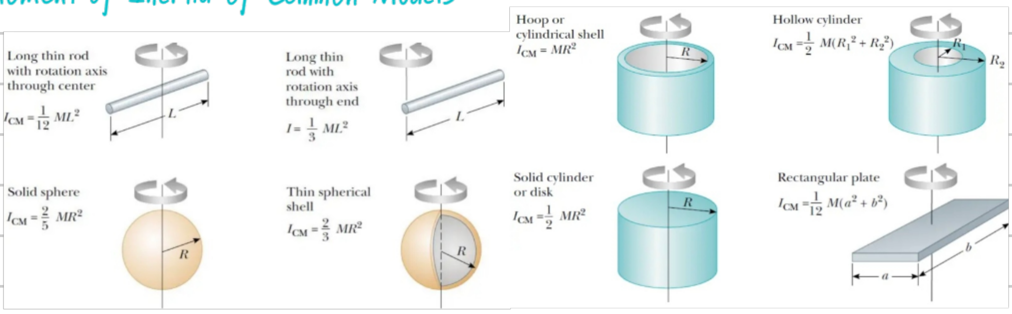
Work - Kinetic Energy Theorem

$\sum\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\cdot\omega$

$\Rightarrow \sum\tau\,d\theta = dW = I\omega\,d\omega$

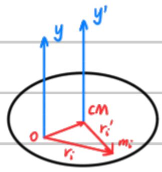
$\Rightarrow \sum W = \int_{\theta_i}^{\theta_f}\sum\tau\,d\theta = \int_{\omega_i}^{\omega_f} I\omega\,d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$

Moment of Inertia of Common Models



The Parrallel-Axis Theorem

$I_o = I_{cm} + Mh^2$



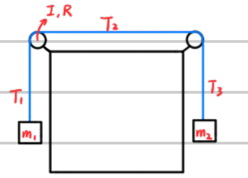
Proof $I_y = \sum m_i r_i^2 = \sum m_i(\vec{r_i}' + \vec{h})^2$

$= \sum m_i r_i'^2 + h^2\sum m_i + 2\sum m_i \vec{r_i}'\cdot\vec{h}$

$= I_{y'} + h^2 M$

Lecture IX - Vector Cross Product ; Coriolis Force

Atwood's Machine



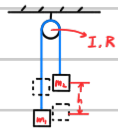
$(T_1 - T_2)R = I\alpha$, $(T_2 - T_3)R = I\alpha$

$m_1 g - T_1 = m_1 a$, $T_3 - m_2 g = m_2 a$

$a = \alpha R$

Solving equations simultaneously yields : $a = \dfrac{(m_1 - m_2)g}{m_1 + m_2 + \frac{2I}{R^2}}$

Connected Cylinders



$\Delta K + \Delta U_1 + \Delta U_2 = 0$

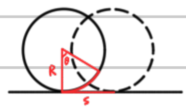
$\Delta K = (\frac{1}{2} m_1 v_f^2 + \frac{1}{2} m_2 v_f^2 + \frac{1}{2} I \omega_f^2)$

$v_f = \omega_f R$

$\Delta U_1 = m_1 g h$, $\Delta U_2 = -m_2 g h$

Solving equations simultaneously yields : $v_f = \sqrt{\dfrac{2(m_2 - m_1)gh}{m_1 + m_2 + \frac{I}{R^2}}}$

Pure Rolling Motion

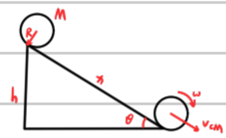


$s = \theta R$

$v_{cm} = \frac{ds}{dt} = \frac{d\theta}{dt} R = \omega R$

$a_{cm} = \frac{dv_{cm}}{dt} = \frac{d\omega}{dt} R = \alpha R$

Rolling Sphere



$I = \frac{2}{5} M R^2$

$\Delta K + \Delta U = 0$

$\Delta K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2$

$\Delta U = -Mgh$

$v_{cm} = \omega R$

Solving equations simultaneously yields : $v_{cm} = \sqrt{\frac{10}{7} gh}$

Derivative of the Cross Product

$\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$

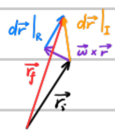
Coliolis Effect

The Coriolis effect is a deflection of moving objects when the motion is described relative to a rotating reference frame.

In the inertia frame of reference : $\left.\frac{d\vec{r}}{dt}\right|_I = \left.\frac{d\vec{r}}{dt}\right|_R + \vec{\omega} \times \vec{r}$

change of the displacement within the rotational frame

change of the displacement due to the rotation of the reference frame



Acceleration under Colionis Effect

$\vec{v} = \left.\frac{d\vec{r}}{dt}\right|_I = \left.\frac{d\vec{r}}{dt}\right|_R + \vec{\omega} \times \vec{r}$

$\vec{a} = \left.\frac{d\vec{v}}{dt}\right|_I = \left.\frac{d^2\vec{r}}{dt^2}\right|_I = \left[\left.\frac{d}{dt}\right|_R + \vec{\omega}\right] \times \vec{r} = \left.\frac{d^2\vec{r}}{dt^2}\right|_R + 2\vec{\omega} \times \left.\frac{d\vec{r}}{dt}\right|_R + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

'Force' in the Rotating Frame

$$\begin{aligned} F_{net} &= m \left.\frac{d^2\vec{r}}{dt^2}\right|_R \\ &= m \left.\frac{d^2\vec{r}}{dt^2}\right|_I - 2m\vec{\omega} \times \left.\frac{d\vec{r}}{dt}\right|_R - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= F - 2m\vec{\omega} \times \vec{v}_R + m\omega^2 \vec{r} \end{aligned}$$

Proof $m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(m\vec{\omega} \cdot \vec{r}) - \vec{r}(m\vec{\omega} \cdot \vec{\omega}) = -m\omega^2 \vec{r}$ $\quad$ $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Note We usually name $-2m\vec{\omega} \times \vec{v}_R$ as Coriolis force, and name $m\omega^2\vec{r}$ as centrifugal force

Lecture X - Rolling Motion and Angular Momentum

Angular Momentum

$\sum\vec{F}=\frac{d\vec{p}}{dt}$

$\sum\vec{\tau}=\vec{r}\times\sum\vec{F}=\vec{r}\times\frac{d\vec{p}}{dt}$

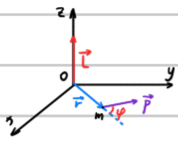
$\frac{d}{dt}(\vec{r}\times\vec{p})=\vec{r}\times\frac{d\vec{p}}{dt}+\frac{d\vec{r}}{dt}\times\vec{p}=\vec{r}\times\frac{d\vec{p}}{dt}+\vec{v}\times(m\vec{v})=\vec{r}\times\frac{d\vec{p}}{dt}$

$\sum\vec{\tau}=\frac{d(\vec{r}\times\vec{p})}{dt}$

Hence, we define the angular momentum $\vec{L}=\vec{r}\times\vec{p}$, and $\sum\vec{\tau}=\frac{d\vec{L}}{dt}$

Angular Momentum Properties

$L=mvr\sin\varphi$



Given that $\vec{r}$ changes with the choice of the coordinate origin, the angular momentum depends on the choice of the coordinate origin as well.

For a system of particles, $\vec{L}=\sum_i\vec{L}_i$, $\sum\vec{\tau}_{ext}=\sum_i\frac{dL_i}{dt}=\frac{d}{dt}\sum_i L_i=\frac{d\vec{L}}{dt}$

$\vec{L}=\sum\vec{r}_i\times m_i\vec{v}_i=\sum\vec{r}_i\times m_i(\vec{\omega}\times\vec{r}_i)=\sum\left[m_i r_i^2\vec{\omega}-m_i\vec{r}_i(\vec{r}_i\cdot\vec{\omega})\right]$

Axial Angular Momentum

$\vec{L}_{axis}=\sum_i m_i r_i^2\omega=\left(\sum_i m_i r_i^2\right)\omega=I\omega$

$\sum\tau_{ext}=\frac{dL_{axis}}{dt}=I\alpha$

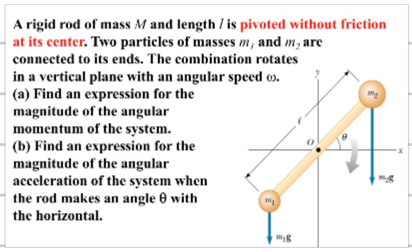
Two Connected Particles Rotation

$I=\frac{1}{12}Ml^2+m_1\left(\frac{l}{2}\right)^2+m_2\left(\frac{l}{2}\right)^2=\frac{l^2}{4}\left(\frac{M}{3}+m_1+m_2\right)$

$L=I\omega=\frac{l^2}{4}\left(\frac{M}{3}+m_1+m_2\right)\omega$

$\sum\tau_{ext}=\tau_1+\tau_2=m_1g\frac{l}{2}\cos\theta-m_2g\frac{l}{2}\cos\theta=\frac{1}{2}(m_1-m_2)gl\cos\theta$

$\alpha=\frac{\sum\tau_{ext}}{I}=\frac{2(m_1-m_2)g\cos\theta}{l\left(\frac{M}{3}+m_1+m_2\right)}$



Conservation of Angular Momentum

When the resultant external torque acting on the system is zero, $\sum\tau_{ext}=\frac{d\vec{L}}{dt}=0\Rightarrow\vec{L}=\text{constant}$

This property is called the conservation of angular momentum.

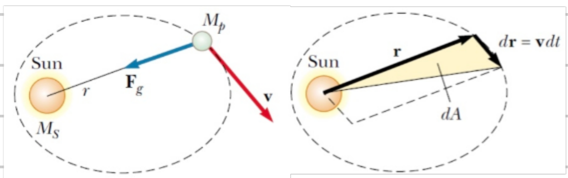
Kepler's Second Law

$\vec{\tau}=\vec{r}\times\vec{F}=0$

$\vec{L}=\vec{r}\times\vec{p}=\vec{r}\times M_p\vec{v}=M_p\vec{r}\times\vec{v}=\text{constant}$

$dA=\frac{1}{2}|\vec{r}\times d\vec{r}|=\frac{1}{2}|\vec{r}\times\vec{v}\,dt|=\frac{L}{2M_p}dt$

$\frac{dA}{dt}=\frac{L}{2M_p}=\text{constant}$



Isolated Systems

For an isolated system:

- $K_i+U_i=K_f+U_f$
- $\vec{p}_i=\vec{p}_f$
- $\vec{L}_i=\vec{L}_f$

Lecture XI - Simple Harmonic Motion

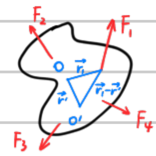
Necessary conditions for equilibrium

- The net force acting on an object is zero (Translational equilibration)
- The net torque about any axis is zero (Rotational equilibration)

Static Equilibration

- An object is in equilibration: $a=0,\ \alpha=0$
- An object is in stable equilibration: $a=0,\ \alpha=0,\ v=0,\ \omega=0$

Torque about Any Point



$$\sum \vec{\tau_o} = \sum_i \vec{r_i} \times \vec{F_i}$$

$$\sum \vec{\tau_{o'}} = \sum_i (\vec{r_i} - \vec{r'}) \times \vec{F_i} = \sum_i \vec{r_i} \times \vec{F_i} - \vec{r'} \times \sum_i \vec{F_i} = \sum \vec{\tau_o} - \vec{r'} \times \sum_i \vec{F_i}$$

Note That means if an object is in translational equilibrium and the net torque is zero about one point, then the net torque must be zero about any other point.

Simple Harmonic Motion

An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directly.

$$ma = -kx$$

Quantitative Analysis

Define $\omega = \sqrt{\frac{k}{m}}$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

$\Rightarrow x = A\cos(\omega t + \varphi)$ A: Amplitude of the oscillation, $T = \frac{2\pi}{\omega}$: Period of the oscillation, φ: Phase angle

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Energy of the Harmonic Oscillator

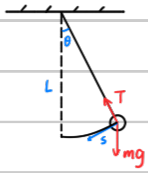
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \varphi)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \varphi)$$

$$E = K + U = \frac{1}{2}kA^2$$

Lecture XII - Applications of Oscillation Motion

Simple Pendulum



$\sum F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$, $s = L\theta$

$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta$

$\Rightarrow \theta = \theta_{max}\cos(\omega t + \varphi)$, $\omega = \sqrt{\frac{g}{L}}$

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$

Physical Pendulum

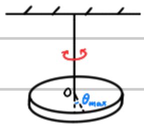
If a hanging object oscillates about a fixed axis that does not pass through its centre of mass and the object cannot be approximated as a point mass, the system is called a physical pendulum.

$-mgd\sin\theta = I\frac{d^2\theta}{dt^2} \Rightarrow \omega = \sqrt{\frac{mgd}{I}}$

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$

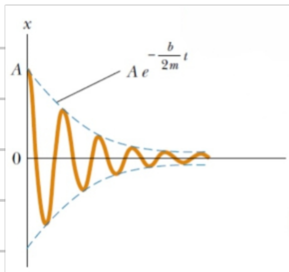
Note We cannot treat the system as a simple pendulum.

Torsional Pendulum



$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$

Damped Oscillator



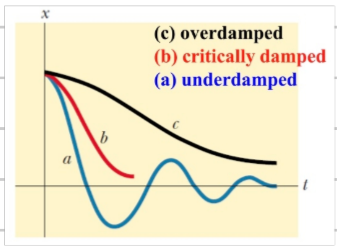
Define b: damping coefficient

$\sum F_x = -kx - bv = ma_x$

$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \varphi)$, $\omega = \sqrt{\frac{k}{m} - (\frac{b}{2m})^2}$

Critical Damping



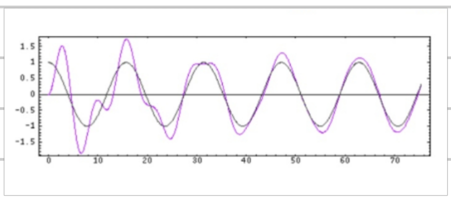
For critical damping, $\omega = 0 \Rightarrow \frac{k}{m} = (\frac{b}{2m})^2$

Forced Oscillation

$F_{ext}\cos\omega t - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$

$x = \underbrace{A'e^{-\frac{b}{2m}t}\cos(\omega' t + \varphi)}_{\text{Transient solution}} + \underbrace{A\cos(\omega t + \varphi)}_{\text{Steady solution}}$, $\omega' = \sqrt{\frac{k}{m} - (\frac{b}{2m})^2}$

Slow Drive

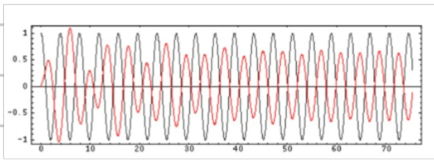


$v \to 0$, $a \to 0 \Rightarrow F_{ext}\cos\omega t - kx = 0$

The driving force is slow enough that the oscillator can follow the force after the transient motion decays.

$\omega < \omega_0 = \sqrt{\frac{k}{m}}$

Fast Drive

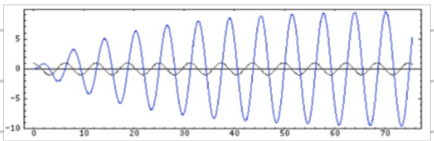


$x \to 0,\ v \to 0 \Rightarrow F_{ext}\cos\omega t = m\frac{d^2x}{dt^2}$

The driving force is slow enough that the oscillator cannot follow the force and lags behind (π out of phase). Note that the amplitude is smaller than that for slow drive.

$\omega > \omega_0 = \sqrt{\frac{k}{m}}$

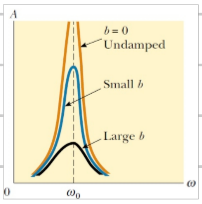
At Resonance



$x \to 0,\ a \to 0 \Rightarrow F_{ext}\cos\omega t - b\frac{dx}{dt} = 0$

The amplitude quickly grows to a maximum. After the transient motion decays and the oscillator settles into steady state motion, the displacement $\frac{\pi}{2}$ out of phase with force (displacement lags the force).

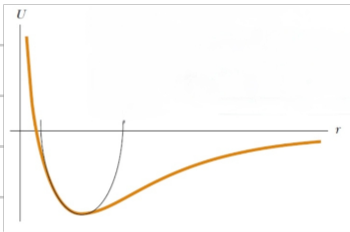
Forced Oscillation



Steady state: $x = A\cos(\omega t + \varphi)$

$A = \frac{\frac{F_{ext}}{m}}{\sqrt{(\omega^2-\omega_0^2)^2 + (\frac{b\omega}{m})^2}}$, $\omega_0 = \sqrt{\frac{k}{m}}$

The Equilibrium



Lennard-Jones Potential: $U(x) = 4\varepsilon\left[(\frac{\sigma}{x})^{12} - (\frac{\sigma}{x})^6\right]$

$\frac{dU}{dx} = 0 \Rightarrow x_0 = 2^{\frac{1}{6}}\sigma$

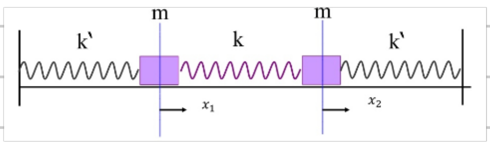
Force Near the Equilibrium

$U(x) = 4\varepsilon\left[(\frac{\sigma}{x})^{12} - (\frac{\sigma}{x})^6\right] = \varepsilon\left[(\frac{x_0}{x})^{12} - 2(\frac{x_0}{x})^6\right]$

$F(x) = -\frac{dU(x)}{dx} = \frac{12\varepsilon}{x_0}\left[(\frac{x_0}{x})^{13} - (\frac{x_0}{x})^7\right] = -\frac{d^2U}{dx^2}\Big|_{x=x_0}(x-x_0) + O((x-x_0)^2) = -\frac{72\varepsilon}{x_0^2}(x-x_0)$

Effective spring constant: $k = \frac{72\varepsilon}{x_0^2} \Rightarrow \omega = \sqrt{\frac{72\varepsilon}{\mu x_0^2}}$

Two Harmonic Oscillators



$m\frac{d^2x_1}{dt^2} = -k'x_1 - k(x_1 - x_2)$, $m\frac{d^2x_2}{dt^2} = -k'x_2 - k(x_2 - x_1)$

$\Rightarrow m\omega^2 x_{10} = (k'+k)x_{10} - kx_{20}$, $m\omega^2 x_{20} = -kx_{10} + (k'+k)x_{20}$

$\Rightarrow \begin{pmatrix} k'+k-m\omega^2 & -k \\ -k & k'+k-m\omega^2 \end{pmatrix}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = 0$

Solution 1: $\omega = \sqrt{\frac{k'+2k}{m}} \to \sqrt{\frac{k}{\frac{m}{2}}}$ $(k' \to 0)$

$k' \to 0 \Rightarrow x_{10} = -x_{20}$ Vibration with the reduced mass

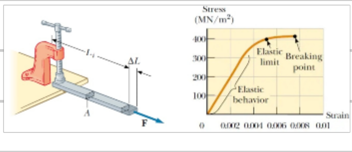
Solution 2: $\omega = \sqrt{\frac{k'}{m}} \to 0$ $(k' \to 0)$

Here, $x_{10} = x_{20}$ Transition

Elastic Properties

Elastic modulus $= \frac{\text{stress}}{\text{strain}}$ for sufficiently small stresses

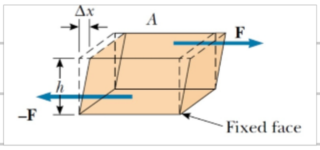
Elasticity of Length



Young's Modulus: $Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{\frac{F}{A}}{\frac{\Delta L}{L_i}} = \frac{FL_i}{A(\Delta L)}$

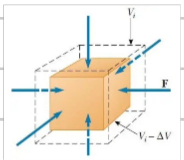
Elasticity of Shape

Shear Modulus: $S = \dfrac{\text{shear stress}}{\text{shear strain}} = \dfrac{\frac{F}{A}}{\frac{\Delta x}{h}} = \dfrac{Fh}{A(\Delta x)}$



Elasticity in Volume

Bulk Modulus: $B = \dfrac{\text{volume stress}}{\text{volume strain}} = \dfrac{\frac{F}{A}}{\frac{\Delta V}{V_i}} = \dfrac{FV_i}{A(\Delta V)}$



Mode Counting

- N-atom linear molecule:

  3 Translation modes, 2 Rotation modes, 3N-5 Vibration modes

- N-atom nonlinear molecule:

  3 Translation modes, 3 Rotation modes, 3N-5 Vibration modes

Lecture XIII - Wave Motion

Basic Varibles of Wave Motion

- Wavelength λ
- Period T
- Frequency f
- Amplitude A

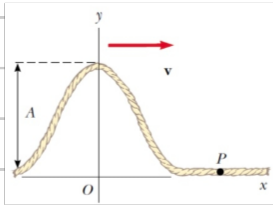
Longitudinal and Transverse Wave

- Longitudinal wave: A wave that causes the particles of the medium to move parallel to the direction of wave motion.
- Transverse wave: A wave that causes the particles of the disturbed medium to move perpendicular to the wave motion.

One Dimension Traveling Waves

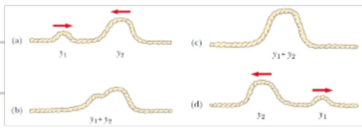
$y = f(x - vt)$

Here, v is the wave speed



Superposition Principle

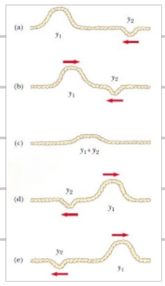
If two or more traveling waves are moving through a medium, the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves.



Interference

The combination of seperate waves in the same region of space to produce a resultant wave is called interference.

- Constructive interference: The resultant wave has greater amplitude
- Destructive interference: The resultant wave has lower amplitude

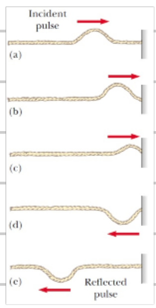


Reflection

When the pulse reaches the support, a severe change in the medium occurs.

The wave undergoes refletion. that is, the pulse moves back along the string in the opposite direction.

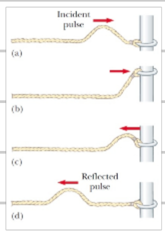
By Newton's third law, the support must exert an equal and opposite reaction force on the string. This force causes the pulse to invert upon reflection.



Free Boundary Condition

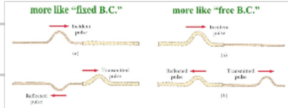
The tension at the free end is maintained.

Then, the pulse is reflected, but this time it is not inverted.



Transmission

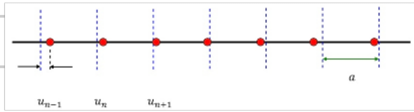
When the boundary is intermediate between two extremes, part of the incident pulse is reflected and part goes transmission, that is, some of the pulse passes through the boundary.



The Linear Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \ , \quad v = a\sqrt{\frac{K}{M}}$$

Proof Consider an example in one dimension, then



Equilibrium positions: $X_n = na$

Deviations from the equilibrium: $u_n = x_n - X_n$

Then, $u_{n+1} - u_n = [x_{n+1} - (n+1)a] - [x_n - na] = x_{n+1} - x_n - a$

Consider nearest-neighbour interactions only: $\varphi(x_{n+1} - x_n) = \varphi_0 + \frac{1}{2}K(x_{n+1} - x_n - a)^2 + \cdots$

$$= \varphi_0 + \frac{1}{2}K(u_{n+1} - u_n)^2 + \cdots$$

In the harmonic approximation, $U_{harm} = \frac{1}{2}K\sum_n (u_{n+1} - u_n)^2$

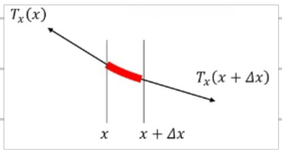
$$M\ddot{u}_n = -\frac{dU_{harm}}{du_n} = K[u_{n+1} - u_n] - K[u_n - u_{n-1}] = (Ka)\left.\frac{\Delta u}{\Delta x}\right|_{x_n + \frac{a}{2}} - (Ka)\left.\frac{\Delta u}{\Delta x}\right|_{x_n - \frac{a}{2}}$$

Note $(Ka)\left.\frac{\Delta u}{\Delta x}\right|_{x_n + \frac{a}{2}} = \frac{u_{n+1} - u_n}{a} \longrightarrow \left.\frac{\partial u}{\partial x}\right|_{x_n + \frac{a}{2}} \quad (\Delta x \to 0)$

Hence, for wavelength much greater than a, $M\left(\frac{\partial^2 u}{\partial t^2}\right)_{x_n} = (Ka)\left[\left(\frac{\partial u}{\partial x}\right)_{x_n + \frac{a}{2}} - \left(\frac{\partial u}{\partial x}\right)_{x_n - \frac{a}{2}}\right] = (Ka^2)\left(\frac{\partial^2 u}{\partial x^2}\right)_{x_n}$

Lecture XIV - Sinusoidal Waves

The Speed of Waves on Strings



$\Delta m = \mu(\Delta x)$

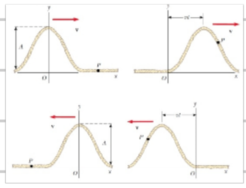
$(\Delta m)\dfrac{\partial^2 y}{\partial t^2} = \mu(\Delta x)\dfrac{\partial^2 y}{\partial t^2} = T_x(x+\Delta x)\dfrac{\partial y}{\partial x}\Big|_{x+\Delta x} - T_x(x)\dfrac{\partial y}{\partial x}\Big|_x$

$T_x(x+\Delta x) = T_x(x) = T$

$\Rightarrow \dfrac{\partial^2 y}{\partial t^2} = \dfrac{T}{\mu}\dfrac{\partial^2 y}{\partial x^2}$

$\Rightarrow v = \sqrt{\dfrac{T}{\mu}}$

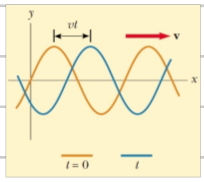
General Solutions



The linear wave equation: $\dfrac{\partial^2 y}{\partial t^2} = v^2\dfrac{\partial^2 y}{\partial x^2}$

Wave functions: $y = f(x+vt)$, $y = f(x-vt)$

Sinusoidal Waves



For the linear wave equation: $\dfrac{\partial^2 y}{\partial t^2} = v^2\dfrac{\partial^2 y}{\partial x^2}$

The most important family of the solutions are: $y = A\sin(kx - \omega t + \varphi)$

Here, $\omega = vk$

k: angular wave number

ω: angular frequency

φ: phase constant

Various Forms

$y = A\sin(kx - \omega t) \xrightarrow{t=0} y = A\sin\left(\frac{2\pi}{\lambda}x\right)$

$\omega = \frac{2\pi}{T} \rightarrow f = \frac{1}{T}$

$\omega = \frac{2\pi}{T},\ k = \frac{2\pi}{\lambda} \xrightarrow{} v = \frac{\omega}{k}$

$f = \frac{1}{T} \xrightarrow{v = \lambda f} v = \frac{\lambda}{T}$

$k = \frac{2\pi}{\lambda} \rightarrow v = \frac{\lambda}{T}$

$y = A\sin(kx - \omega t) \rightarrow y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \rightarrow y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \xrightarrow{t=0} y = A\sin\left(\frac{2\pi}{\lambda}x\right)$

Sinusoidal Wave on Strings

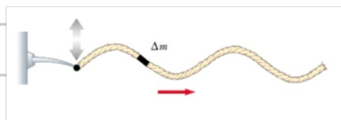
Each segment oscillates in the y direction.

$y = A\sin(kx - \omega t)$

$v_y = \dfrac{dy}{dt}\Big|_{x=\text{constant}} = \dfrac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$

$a_y = \dfrac{dv_y}{dt}\Big|_{x=\text{constant}} = \dfrac{\partial v_y}{\partial t} = -\omega^2 A\sin(kx - \omega t)$

Rate of Energy Transfer



$\Delta U = \frac{1}{2}(\Delta m)\omega^2 y^2 = \frac{1}{2}(\mu\Delta x)\omega^2 y^2$

$dU = \frac{1}{2}\mu\omega^2[A\sin(kx - \omega t)]^2 dx = \frac{1}{2}\mu\omega^2 A^2\sin^2(kx - \omega t)\,dx$

For simple harmonic oscillation, the total energy $E = K + U$ is constant, i.e., $dE = \frac{1}{2}\mu\omega^2 A^2 dx$

The rate of energy transfer: $P = \frac{dE}{dt} = \frac{1}{2}\mu\omega^2 A^2 v$

The Principle of Superposition

the resultant wave

When two or more waves move in the same linear medium, the net displacement of the medium at any point equals the algebraic sum of all the displacements caused by the individual waves.

Interference

Same frequency, wavelength, amplitude, direction. Different phase

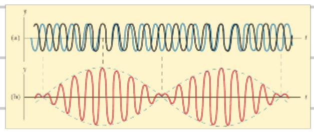
$y_1 = A\sin(kx-\omega t)$, $y_2 = A\sin(kx-\omega t+\varphi)$

$y = y_1+y_2 = 2A\cos\frac{\varphi}{2}\sin(kx-\omega t+\frac{\varphi}{2})$

- When $\cos\frac{\varphi}{2}=\pm1$ ($\varphi=2k\pi$), the waves are said to be everywhere in phase and thus interfere constructively.
- When $\cos\frac{\varphi}{2}=0$ ($\varphi=\frac{k+1}{2}\pi$), the resultant wave has zero amplitude everywhere, as a consequence of destructive interference.

Beating: Temporal Interference

Beating is the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.



$y_1 = A\cos\omega_1 t = A\cos 2\pi f_1 t$, $y_2 = A\cos\omega_2 t = A\cos 2\pi f_2 t$

$y = y_1+y_2 = 2A\cos 2\pi(\frac{f_1-f_2}{2})t\ \cos 2\pi(\frac{f_1+f_2}{2})t$

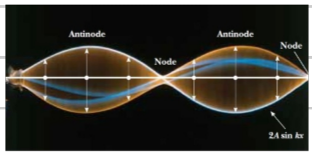
The amplitude and therefore the intensity of the resultant sound vary in time: $A_{resultant} = 2A\cos 2\pi(\frac{f_1-f_2}{2})t$

The two neighbouring maxima in the envelop function are seperated by $2\pi(\frac{f_1-f_2}{2})t=\pi$

Beat frequency: $f_b = |f_1-f_2|$

Standing Waves

Same frequency, wavelength, amplitude. Different direction.



$y_1 = A\sin(kx-\omega t)$, $y_2 = A\sin(kx+\omega t)$

$y = y_1+y_2 = (2A\sin kx)\cos\omega t$

Nodes: $kx = n\pi \Rightarrow x = \frac{k}{2}\pi$ Antinodes: $kx = (n+\frac{1}{2})\pi \Rightarrow x = \frac{2k-1}{4}\pi$

Lecture XV - Sound Waves

Sound Intensity

We define the intensity I of a wave, or the power per unit area, to be the rate at which the energy being transported by the wave flows through a unit area A perpendicular to the direction of travel of the wave.

$I = \frac{P}{A} = \frac{1}{2}\rho v(\omega s_{max})^2$ ρ: density v: speed of sound ω: frequency s_{max}: amplitude

Sound Level

Threshold of hearing: $I_0 = 1.00 \times 10^{-12}\ W \cdot m^{-2}$

$\beta = 10 \log\left(\frac{I}{I_0}\right)$ in decibels (dB)

Speed of Sound in a Solid

$v = \sqrt{\frac{Y}{\rho}}$ Y: Young's modulus for the material ρ: the density of material

Note Another equivalent form is $\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow v = a\sqrt{\frac{K}{M}}$

Speed of Sound in a Fluid

The speed of all mechanical waves follows the general form: $v = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}$

Then, $v = \sqrt{\frac{dp}{d\rho}} = \sqrt{\kappa RT}$

Doppler Effect

When both source and observer are in motion, the observed frequency $f' = \frac{v + v_o}{v - v_s} f$.

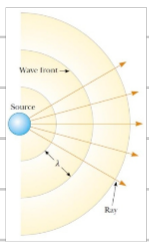
Note Toward → increase in observed frequency

Away from → decrease in observed frequency

Spherical Waves

The wave intensity at a distance r from the source: $I = \frac{P_{avg}}{A} = \frac{P_{avg}}{4\pi r^2}$

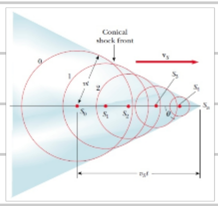
The intensity is proportional to the square of the amplitude. Hence, $\psi(r,t) = \frac{s_0}{r}\sin(kr - \omega t)$



Shock Waves

Mach number: $\frac{v_s}{v}$

$\sin\theta = \frac{vt}{v_s t} = \frac{v}{v_s}$



Lecture XVII – The Speed of Light and the Principles of Relativity

The Speed of Light

$c = 299\ 792\ 458\ m \cdot s^{-1}$ in any medium

Lecture XVIII - Combining Velocities and Synchronizing Clocks

Combining Velocities

- Nonrelativistic velocity addition law : $w = u + v$
- Relativistic velocity addition law : $w = \frac{u+v}{1+(\frac{u}{c})(\frac{v}{c})}$

Lecture XIX - Moving Clocks and Moving Sticks

Slowing-Down Factor

$s = \sqrt{1 - \frac{v^2}{c^2}}$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Time Dilation

$\Delta t = \gamma (\Delta t_0)$

Length Contraction

$L = \gamma L_0$

Lecture XX - Lorentz Transformation

Lorentz Transformation

When the object moving along the x-axis direction:

- $t' = \gamma(t + \frac{vx}{c^2})$
- $x' = \gamma(x + vt)$
- $y' = y$
- $z' = z$

Lecture XXI - Relativistic Energy and Momentum

Linear Momentum

$p = \gamma m v$

Newton's second law: $F = \frac{dp}{dt}$

Relativity Energy

$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx = (\gamma - 1)mc^2$

We define:

- Rest Energy: $E_0 = mc^2$
- Total Energy: $E = \gamma mc^2$

Relativistic Kinetic Energy

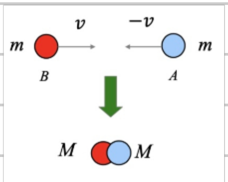
$K = (\gamma - 1)mc^2$

Relativistic Collision I

Total energy before collision: $E_i = 2\gamma mc^2$

Total energy after collision: $E_f = 2Mc^2$

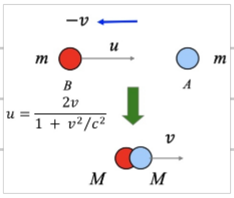
Energy conservation gives $E_i = E_f \Rightarrow M = \gamma m$



Relativistic Collision II

Momentum conservation: $\gamma_u m u = 2\gamma_v M v$

Energy conservation: $\gamma_u mc^2 + mc^2 = 2\gamma_v Mc^2$



Relativistic Doppler Effect

$\lambda = \frac{(c-v)T}{n}$

$n = f_0 s T$

$f = \frac{c}{\lambda} = \frac{s}{1 - \frac{v}{c}} f_0 = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\, f_0$

Lecture XXII - More Twists on Space and Time

Relationship between Energy and Slowing-Down Factor

$\Delta U = \frac{1}{2}mv^2 = mgh$

$s = \sqrt{1-\frac{v^2}{c^2}} = 1 - \frac{1}{2}\frac{v^2}{c^2} = 1 - \frac{\Delta U}{mc^2}$

due to $v \ll c$

Lecture XXIII - Basic Concepts of Thermodynamics

The Zeroth Law of Thermodynamics

If two systems are in thermal equilibrium with a third system, then they must be in thermal equilibrium with each other.

Temperature

Temperature is a measure of the tendency of an object to spontaneously give up energy to its surrondings. When two objects are in thermal contact, the one that tends to spontaneously lose energy is at the higher temperature.

The Third Law of Thermodynamics

It is impossible for any procedure to lead to the isoterm $T=0$ in a finite number of steps.

Temperature Scales

- Celsius scale: $T_C = T - 273.15$
- Fahrenheit scale: $T_F = \frac{9}{5}T_C + 32$

Linear and Volume Expansion

- Linear expansion: $\Delta L = \alpha L_0 (\Delta T)$
- Volume expansion: $\Delta V = \beta V_0 (\Delta T)$

$\beta = 3\alpha$

Proof $L^3 = (L_0 + \Delta L)^3 = [L_0(1+\alpha(\Delta T))]^3 = L_0^3[1+3\alpha(\Delta T) + o(\Delta T)^2]$

$V = V_0(1+\beta(\Delta T))$

$\Rightarrow 3\alpha = \beta$

Low-Density Gases

$pV = nRT = \frac{N}{N_A}RT = Nk_BT$ Boltzmann's constant: $k_B = \frac{R}{N_A} = 1.38\times10^{-23}\ J\cdot K^{-1}$

Real Gases

$(p + \frac{an^2}{V^2})(V - bn) = Nk_BT$ a: due to potential energy, b: volume of a module

Lecture XXIV – Microscopic Model of Ideal Gases

热学 $\begin{cases} \text{热力学 - 宏观} \\ \text{气体动理论 - 微观} \end{cases}$

气体动理论

一、分子运动的基本概念

1. 概念

- 压强：大量分子对器壁碰撞的平均冲量
- 体积：分子活动的空间 （并非分子大小的总和）
- 温度：物体冷热程度的量度，反映分子热运动剧烈程度

2. 平衡状态

- 平衡态：一个孤立系统，宏观状态参量都不随时间变化的状态 （热动平衡）
- 平衡过程：在过程进行的每一时刻，系统都无限接近平衡态，即准静态过程

3. 理想气体状态方程（克拉伯龙方程）

$pV=\frac{m}{M}RT=\nu RT=\nu N_0kT=NkT$ 　m：气体质量，M：气体摩尔质量，N：气体分子数，R：摩尔气体常量，N_0：阿伏伽德罗常数

$p=nkT$ 　n：单位体积分子数，k：玻尔兹曼常数

$\Rightarrow n=\frac{N}{V}$

二、气体分子的热运动

1. 统计规律的特征

- 统计平均值：$\overline{M}=\frac{N_1M_1+N_2M_2+\cdots}{N}$，$N=N_1+N_2+\cdots$

例：$\overline{v_x}=\frac{\sum_i \Delta N_i v_{ix}}{N}=\frac{\int v_x dN}{N}$

$\overline{v_x^2}=\frac{\sum_i \Delta N_i v_{ix}^2}{N}=\frac{\int v_x^2 dN}{N}$

$\overline{\varepsilon}=\frac{\sum_i \frac{1}{2}\mu v_i^2\Delta N_i}{N}=\frac{\int(\frac{1}{2}\mu v^2)dN}{N}$

- 概率：$W_i=\lim\limits_{N\to+\infty}\frac{N_i}{N}$
- 归一化：所有可能出现的状态的概率相加，其和为1.

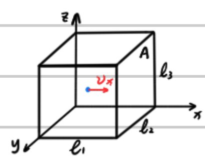
归一化条件：$\sum_i W_i=1$

三、理想气体的压强公式

1. 理想气体的模型

- 微观模型：分子中心间距远大于分子直径；两次碰撞间分子作匀速直线运动；单个分子能量仅含动能，不考虑分子间相互作用势能；弹性碰撞，无能量损失
- 统计假设：平衡态下分子密度均匀分布，作热运动的分子向各方向运动机会均等 （$\overline{v_x}=\overline{v_y}=\overline{v_z}=0$，$\overline{v_x^2}=\overline{v_y^2}=\overline{v_z^2}=\frac{1}{3}\overline{v^2}$）

2. 理想气体的压强公式



一个分子与A面碰撞一次，分子动量改变 $2\mu v_x$，也即分子对A面的冲量

1s内一个分子与A面的碰撞次数：$\frac{v_x}{2l_1}$

1s内N个分子对A面的冲量：$I=\sum_{i=1}^{N}\frac{v_{ix}}{2l_1}\cdot 2\mu v_{ix}$ 　μ：分子质量

对A面压强：$p=\frac{I}{l_2l_3}=\frac{\mu N}{l_1l_2l_3}\sum_{i=1}^{N}\frac{v_{ix}^2}{N}=n\mu\overline{v_x^2}$ 　n：单位体积分子数

由统计假设：$\overline{v_x^2}=\frac{1}{3}\overline{v^2}\Rightarrow p=\frac{1}{3}n\mu\overline{v^2}$

一个分子的平均平动动能：$\overline{\varepsilon}=\frac{1}{2}\mu\overline{v^2}$

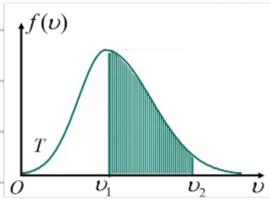
$\Rightarrow p=\frac{2}{3}n\overline{\varepsilon}$

3.理想气体的温度公式

$p=nkT=\frac{2}{3}n\bar{\varepsilon} \Rightarrow \bar{\varepsilon}=\frac{3}{2}kT$ → 表明分子平均平动动能只与温度有关

四、麦克斯韦速率分布律

1.麦克斯韦速率分布曲线



·小条面积：$f(v)dv=\frac{dN}{N}$，表示速率分布在 $v\sim v+dv$ 中的分子比例

·宽条面积：$\int_{v_1}^{v_2} f(v)dv=\frac{\Delta N}{N}$，表示速率分布在 $v_1\sim v_2$ 中的分子比例

·曲线下总面积：$\int_0^{+\infty} f(v)dv=1$，表示速率分布函数的归一化条件

2.麦克斯韦速率分布函数

$$f(v)=4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2kT}}$$

3.应用

·最概然速率 v_p （用于讨论速率分布）

$\frac{df(v_p)}{dv_p}=0\Rightarrow v_p=\sqrt{\frac{2kT}{\mu}}=\sqrt{\frac{2RT}{M}}=1.41\sqrt{\frac{RT}{M}}$ → 温度与 v_p 正相关，分子质量与 v_p 负相关

·平均速率 $\bar{v}$ （用于讨论分子的碰撞次数）

$\bar{v}=\int v\frac{dN}{N}=\int_0^{+\infty} vf(v)dv=\sqrt{\frac{8kT}{\pi\mu}}=\sqrt{\frac{8RT}{\pi M}}=1.59\sqrt{\frac{RT}{M}}$ → 温度与 $\bar{v}$ 正相关，分子质量与 $\bar{v}$ 负相关

·方均根速率 $\sqrt{\overline{v^2}}$ （用于讨论分子的平均平动动能

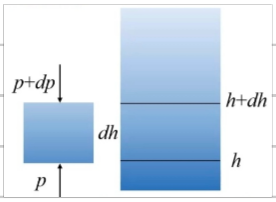
$\overline{v^2}=\int v^2\frac{dN}{N}=\int_0^{+\infty}v^2f(v)dv=\frac{3kT}{\mu}=\frac{3RT}{M}$

$\sqrt{\overline{v^2}}=\sqrt{\frac{3kT}{\mu}}=\sqrt{\frac{3RT}{M}}=1.73\sqrt{\frac{RT}{M}}$ → 温度与 $\sqrt{\overline{v^2}}$ 正相关，分子质量与 $\sqrt{\overline{v^2}}$ 负相关

·$v_p<\bar{v}<\sqrt{\overline{v^2}}$

五、玻尔兹曼分布律

1.重力场中粒子按高度的分布



$dp=-\rho g(dh)=-n\mu g(dh)$

平衡态下气体温度处处相同，气体压强为 $p=nkT\Rightarrow dp=kT(dn)$

$kT(dn)=-n\mu g(dh)\Rightarrow \frac{dn}{n}=-\frac{\mu g}{kT}dh\Rightarrow\int_{n_0}^{n}\frac{dn}{n}=\int_0^h-\frac{\mu g}{kT}dh$

$\Rightarrow n=n_0e^{-\frac{\mu gh}{kT}}$ n_0：$h=0$时的单位体积分子数

①n与h负相关 ②μ越大，n减小越迅速 ③T越大，n减小越缓慢

$p=nkT=n_0kTe^{-\frac{\mu gh}{kT}}=p_0e^{-\frac{\mu gh}{kT}}=p_0e^{-\frac{Mgh}{RT}}$ $p_0=n_0kT$：$h=0$时的压强 （等温气压公式）

2.玻尔兹曼分布律

对任何形式的保守力场，平衡态下温度为T的气体中，位于空间某一区间的分子数为

$N=Ce^{-\frac{\varepsilon}{kT}}$ C：总分子数，ε：该区间中分子的势能 （玻尔兹曼分布律） $e^{-\frac{\varepsilon}{kT}}$：玻尔兹曼因子

①粒子总能量越大，该状态的粒子数越少 ②分子总是优先占据势能较低的状态

六、能量均分原理

1.自由度

自由度 i：确定一个物体在空间位置的独立坐标数

·自由运动质点的自由度（只有平动自由度）

空间自由运动：$i=3$ 平面自由运动：$i=2$ 直线自由运动：$i=1$

·棒的自由度：$i=5$

质心位置：$i_{\text{平}}=3$，棒的方位取向：$i_{\text{转}}=2$

·刚体自由度：$i=6$

质心位置：$i_{\text{平}}=3$，轴的方位取向：$i_{\text{轴}}=2$ 绕轴转动角度：$i_{\text{转}}=1$

· 理想气体分子自由度

单原子分子：$i=3$　　双原子分子：$i=5$　　多原子分子：$i=6$

CO_2 的自由度 $i=5$，因其分子结构为直线型

2. 能量按自由度均分定理

$\bar{\varepsilon}=\frac{1}{2}\mu\overline{v^2}=\frac{1}{2}\mu\overline{v_x^2}+\frac{1}{2}\mu\overline{v_y^2}+\frac{1}{2}\mu\overline{v_z^2}=\frac{3}{2}kT$

$\Rightarrow \frac{1}{2}\mu\overline{v_x^2}=\frac{1}{2}\mu\overline{v_y^2}=\frac{1}{2}\mu\overline{v_z^2}=\frac{1}{2}kT$ → 气体分子平动时，每个自由度上具有相同的热运动能量

温度T的平衡态下，分子每个自由度的平均动能均为 $\frac{1}{2}kT$.（能量自由度均分定理）

对于自由度为 i 的气体分子，该分子的平均总能量为 $\frac{i}{2}kT$

3. 理想气体的内能

理想气体的内能 = 平均动能 + 转动动能

自由度为 i 的 ν mol 理想气体的内能：$E=\frac{m}{M}\cdot\frac{i}{2}RT=\nu\cdot\frac{i}{2}RT$

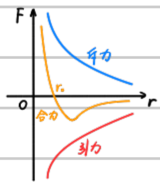
一定量的理想气体的内能只取决于气体分子的自由度 i 和温度 T，而与 p、V 无关.

对同一种气体，ΔT 相同则 ΔE 相同，与具体过程无关.

七. 气体分子的碰撞和平均自由程

1. 分子的有效直径

根据分子力与分子间距离的关系，可将分子碰撞过程划分为3个阶段：



· 合力表现为引力时，分子加速靠近

· 合力为零时，分子仍具有动能

· 合力表现为斥力时，分子减速靠近

因为 $\lim F=+\infty$，故当 $E_k=0$ 时，$r=d>r_0$. d 即称作分子的有效直径.　分子的实际直径

2. 平均碰撞频率、平均自由程

· 碰撞频率 z：一个分子在 1s 内和其它分子碰撞的次数

· 自由程 λ：一个分子连续两次碰撞之间通过的路程

$\bar{\lambda}=\frac{\bar{v}}{\bar{z}}$

$\bar{z}=n\pi d^2\bar{u}$　　n：分子数密度　$\bar{u}$：平均相对运动速度

$\bar{u}=\sqrt{2}\,\bar{v}\Rightarrow\bar{z}=\sqrt{2}\,n\pi d^2\bar{v}$，$\bar{\lambda}=\frac{1}{\sqrt{2}\,n\pi d^2}$

$\bar{v}=\sqrt{\frac{8kT}{\pi\mu}}\Rightarrow\bar{z}=4nd^2\sqrt{\frac{\pi kT}{\mu}}$，$\bar{\lambda}=\frac{kT}{\sqrt{2}\,\pi d^2 p}$

平均自由时间 $\tau=\frac{1}{\bar{z}}$

热力学

一、功、热量、内能、热力学第一定律

1. 内能

内能是一个状态量

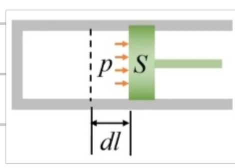
内能：系统中大量分子无规则热运动能量的总和

理想气体的内能：$E=\frac{i}{2}\nu RT=\frac{m}{M}\cdot\frac{i}{2}RT$

2. 作功

功是一个过程量

作功：$dA=Fdl=pSdl=pdV$



· $dV>0$：气体膨胀，对外作正功

· $dV<0$：气体压缩，外界对气体作正功

$A=\int_{V_1}^{V_2} p\,dV$

3. 热量

热是一个过程量

热量：传热过程中，由于温度不同而转移的热运动能量

等压过程：$Q=\frac{m}{M}C_p(T_2-T_1)$　　C_p：等压摩尔热容

等容过程：$Q=\frac{m}{M}C_v(T_2-T_1)$　　C_v：等容摩尔热容

4. 热力学第一定律

$dQ=dE+dA$，$Q=\Delta E+A$

只要求系统初末态为平衡态，过程中经历的各状态不一定为平衡态

二、热力学第一定律对理想气体等值过程的应用

1. 等容过程

$A=p(\Delta V)=0$

$Q_v=\nu C_v(\Delta T)$

$\Delta E=\nu\cdot\frac{i}{2}R(\Delta T)$

$Q_v=\Delta E+A \Rightarrow C_v=\frac{i}{2}R$

2. 等压过程

$A=p(\Delta V)=\nu R(\Delta T)$

$Q_p=\nu C_p(\Delta T)$

$\Delta E=\nu\frac{i}{2}R(\Delta T)=\nu C_v(\Delta T)$

$Q_p=\Delta E+A \Rightarrow C_p=C_v+R=(\frac{i}{2}+1)R$

比热容比：$\gamma=\frac{C_p}{C_v}=\frac{i+2}{i}>1$

3. 等温过程

$\Delta E=0$

$A=\int_{V_1}^{V_2}p\,dV=\int_{V_1}^{V_2}\frac{\nu RT}{V}dV=\nu RT\ln\frac{V_2}{V_1}$

$Q=\Delta E+A=A=\nu RT\ln\frac{V_2}{V_1}$

三、绝热过程

1. 过程方程

$Q=0$，$\Delta E=\nu C_v(\Delta T)$，$A=Q-\Delta E=-\Delta E=-\nu C_v(\Delta T)$

考察微元：$dA+dE=0 \Rightarrow p\,dV=-\nu C_v\,dT$

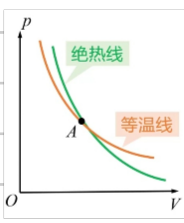
$pV=\nu RT \Rightarrow p\,dV+V\,dp=\nu R\,dT$

$\Rightarrow (C_v+R)p\,dV+C_vV\,dp=0 \Rightarrow \frac{dp}{p}+\gamma\frac{dV}{V}=0$ 或 $pV^\gamma=C_1$　　γ：比热容比

$\Rightarrow TV^{\gamma-1}=C_2$，$p^{\gamma-1}T^{-\gamma}=C_3$

2. 过程曲线

$$\left.\begin{aligned}&\cdot\text{绝热：} pV^\gamma=C_1 \Rightarrow \frac{dp}{dV}=-\gamma\frac{p}{V}\\&\cdot\text{等温：} pV=C_4 \Rightarrow \frac{dp}{dV}=-\frac{p}{V}\end{aligned}\right\}\ \gamma>1 \Rightarrow \text{绝热线下降率比等温线快}$$



$p=nkT \Rightarrow$ 等温过程中，dp是由体积压缩引起的

绝热过程中，dp是由体积压缩和温度升高共同引起的

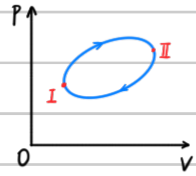
3. 绝热过程功的计算

$A=-\Delta E=-\nu C_v(\Delta T)$

$A=\int_{V_1}^{V_2}p\,dV=\int_{V_1}^{V_2}p_1V_1^\gamma\frac{dV}{V^\gamma}=\frac{1}{\gamma-1}(p_1V_1-p_2V_2)=-\frac{\nu R}{\gamma-1}(\Delta T)$

四、循环过程

1.循环



如果物质系统的状态经历一系列的变化后，又回到了原状态，就称系统经历了一个循环过程.

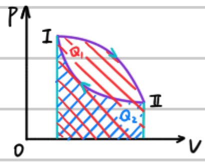
如果循环是准静态过程，在p-V图就构成一闭合曲线.

$\Delta E=0$

$A=\oint dA=$闭合曲线包围的面积（顺时针积分为正值，逆时针积分为负值）

系统对外所作的净功

2.正循环、逆循环

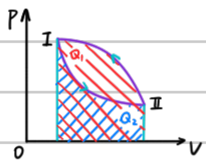


·正循环：循环沿顺时针方向进行

$A=|A_1|-|A_2|=Q_1-Q_2>0$

应用：热机（将热转化为功的装置）

$\eta=\frac{A}{Q_1}=\frac{Q_1-Q_2}{Q_1}=1-\frac{Q_2}{Q_1}<1$



·逆循环：循环沿逆时针方向进行

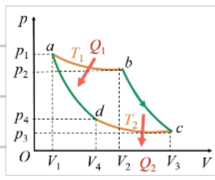
$A=|A_2|-|A_1|=Q_2-Q_1<0$

应用：致冷机（由外界作功，将热量从低温热源送至高温热源，从而使低温热源的温度降低的装置）

$w=\frac{Q_2}{|A|}=\frac{Q_2}{Q_1-Q_2}$

3.卡诺循环：由两个等温过程和两个绝热过程组成的循环过程

·卡诺热机



$a\to b$，$c\to d$：等温线；$b\to c$，$d\to a$：绝热线

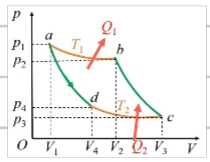
气体从高温热源吸收的热量 $Q_1=\nu RT_1\ln\frac{V_2}{V_1}$

气体向低温热源放出的热量 $Q_2=\nu RT_2\ln\frac{V_3}{V_4}$

绝热过程方程：$T_1V_2^{\gamma-1}=T_2V_3^{\gamma-1}$，$T_2V_4^{\gamma-1}=T_1V_1^{\gamma-1}$

$\Rightarrow\frac{V_2}{V_1}=\frac{V_3}{V_4}\Rightarrow\eta=1-\frac{Q_2}{Q_1}=1-\frac{T_2}{T_1}$ ⇒ η只和T_1、T_2有关

·卡诺制冷机



$a\to b$，$c\to d$：等温线；$b\to c$，$d\to a$：绝热线

气体向高温热源放出的热量 $Q_1=\nu RT_1\ln\frac{V_2}{V_1}$

气体从低温热源吸收的热量 $Q_2=\nu RT_2\ln\frac{V_3}{V_4}$

绝热过程方程：$T_1V_2^{\gamma-1}=T_2V_3^{\gamma-1}$，$T_2V_4^{\gamma-1}=T_1V_1^{\gamma-1}$

$\Rightarrow\frac{V_2}{V_1}=\frac{V_3}{V_4}\Rightarrow w=\frac{Q_2}{Q_1-Q_2}=\frac{T_2}{T_1-T_2}$ ⇒ w只和T_1、T_2有关

4.卡诺定理

·在温度分别为T_1，T_2两个给定热源之间工作的一切可逆热机，其效率相同，都等于理想气体可逆卡诺热机的效率，即$\eta=1-\frac{T_2}{T_1}$

·在相同的高、低温热源之间工作的一切不可逆热机，其效率都不大于可逆热机的效率.

五、热力学第二定律

1.开尔文表述

·不可能只从单一热源吸收热量，使之完全转化为功而不引起其它变化. ⇒ $\eta=\frac{A}{Q_1}<1$

2.克劳修斯表述

·不可能使热量从低温物体传到高温物体而不引起其它变化. ⇒ $\lim w\neq+\infty$

3. 能流图

- 热机：

$$\begin{tikzcd}[row sep=small] \text{高温热源 } T_1 \arrow[d, "Q_1"] \\ \text{热机} \arrow[r] \arrow[d, "Q_2"] & A \\ \text{低温热源 } T_2 \end{tikzcd}$$

- 致冷机：

$$\begin{tikzcd}[row sep=small] \text{高温热源 } T_1 \\ \text{致冷机} \arrow[u, "Q_1"'] & A \arrow[l] \\ \text{低温热源 } T_2 \arrow[u, "Q_2"'] \end{tikzcd}$$

六、可逆过程与不可逆过程

1. 概念

- 可逆过程：若系统经历了一个过程，而过程的每一步都可沿相反的方向进行，同时不引起外界的任何变化，那么这个过程就称作可逆过程
- 不可逆过程：如对于某一过程，用任何方法都不能使系统和外界恢复到原来状态，该过程就是不可逆过程
- 自发过程：不受外界影响而能够自动发生的过程 （单方向进行的不可逆过程）

2. 热力学第二定律的实质

- 自然界的一切自发过程都是单方向进行的不可逆过程 （宏观意义）

七、热力学第二定律的统计学意义

（参考普化）

狭义相对论

一、经典力学的相对性原理

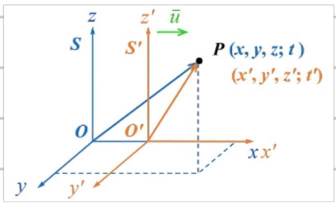
1.伽利略变换

正变换：$x'=x-ut$，$y'=y$，$z'=z$，$t'=t$

$v_x'=v_x-u$，$v_y'=v_y$，$v_z'=v_z$

$a_x'=a_x$，$a_y'=a_y$，$a_z'=a_z$

逆变换：$x=x'+ut$，$y'=y$，$z'=z$，$t'=t$



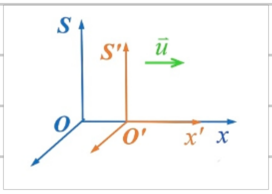
二、狭义相对论原理与洛伦兹变换

1.狭义相对论基本原理

- 光速不变原理：光速不随观察者的运动而变化；光速不随光源的运动而变化；光速不随参考系的运动而变化
- 相对性原理：一切物理规律在所有惯性系中具有相同的形式；所有惯性系都完全处于平等地位

2.洛伦兹变换

(1)空间，时间



正变换：$x'=\gamma(x-ut)$，$y'=y$，$z'=z$，$t'=\gamma(t-\frac{ux}{c^2})$

逆变换：$x=\gamma(x'+ut')$，$y=y'$，$z=z'$，$t=\gamma(t'+\frac{ux'}{c^2})$

$\gamma=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{1}{\sqrt{1-\beta^2}}$，$\beta=\frac{u}{c}$

证：$x=ct$，$x'=ct'$（光速不变原理）

$xx'=\gamma^2(x'+ut)(x-ut)$

联立解得 $\gamma=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

(2)速度

正变换：$v_x'=\frac{v_x-u}{1-\frac{uv_x}{c^2}}$，$v_y'=\frac{v_y}{\gamma(1-\frac{uv_x}{c^2})}$，$v_z'=\frac{v_z}{\gamma(1-\frac{uv_x}{c^2})}$

逆变换：$v_x=\frac{v_x+u}{1+\frac{uv_x}{c^2}}$，$v_y=\frac{v_y'}{\gamma(1+\frac{uv_x}{c^2})}$，$v_z=\frac{v_z'}{\gamma(1+\frac{uv_x}{c^2})}$

证：$v_x'=\frac{dx'}{dt'}=\frac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$，$v_y'=\frac{dy'}{dt'}=\frac{\frac{dy'}{dt}}{\frac{dt'}{dt}}$，$v_z'=\frac{dz'}{dt'}=\frac{\frac{dz'}{dt}}{\frac{dt'}{dt}}$

$v_x=\frac{dx}{dt}=\frac{\frac{dx}{dt'}}{\frac{dt}{dt'}}$，$v_y=\frac{dy}{dt}=\frac{\frac{dy}{dt'}}{\frac{dt}{dt'}}$，$v_z=\frac{dz}{dt}=\frac{\frac{dz}{dt'}}{\frac{dt}{dt'}}$

三、狭义相对论的时空观

1.同时性的相对性：事件的同时性因参照系的选择而异，其为光速不变原理的直接结果

- 同地同时的两个事件在任何惯性系下都是同时的.
- 有因果联系的两事件时序不会颠倒.

2.时间延缓

在某惯性系中，若两事件发生在同一地点，则两事件间的时间间隔称作原时(固有时间)，记作τ_0；其余惯性系中，两事件间的时间间隔记作τ.

$\tau=\gamma\tau_0$

$\tau>\tau_0$ 原时最短

3.长度收缩

在某惯性系中，若某物体静止，则将其在该参考系中测得的长度称作原长，记作l_0；其余惯性系中，同时测量的物体长度记作l.

$l=\frac{l_0}{\gamma}$

$l<l_0$ 原长最长

四、狭义相对论的动力学

1. 质速关系

设 m_0 为质点静止时的质量，即静止质量.

运动质量 $m=\gamma m_0$

2. 相对论动量：$\vec{p}=m\vec{v}=\gamma m_0\vec{v}$

3. 相对论力学

$$\vec{F}=\frac{d\vec{p}}{dt}=\frac{dm}{dt}\vec{v}+m\frac{d\vec{v}}{dt}$$

4. 相对论动能：$E_k=mc^2-m_0c^2=(\gamma-1)m_0c^2$

其中，称 m_0c^2 为静止能量，mc^2 为总能量

5. 质能关系：$E=mc^2$　E：相对论能量，m：相对论质量

6. 相对论能量和动量的关系：$E^2=p^2c^2+E_0^2$

记 $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow m^2c^4=m^2v^2c^2+m_0^2c^2\Rightarrow E^2=p^2c^2+E_0^2$